# Comments on Test 1:

## Test Corrections:

① Redo questions on a separate piece of paper where you lost points

② Explain in sentences the math mistakes made.

③ If both ① & ② are correct for a given question, I will give you half the points back on that question.

④ You can do this as many times as you want with a final due date of Nov 2.

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## Comments on Cross Product, Parallel vectors:

① It is true that if two vectors $V$ & $W$ are parallel, then $V \times W = (0,0,0)$.
(in $\mathbb{R}^3$)

② However, you probably should

use this fact to determine if two vectors are parallel.

Why? ⓐ Cross product takes a long time.

ⓑ It only works for vectors in $\mathbb{R}^3$.

③ How should you determine if two vectors are parallel?

$V$ & $W$ are parallel,

$\Longleftrightarrow$ $V = cW$ or $W = cV$

constant

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Example. Are these vectors parallel?

① $(1,3,-2), (7,0,8)$

② $(1,2,6,-1), (-2,-4,-12,2)$

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① $c(1,3,-2) = (7,0,8)$

then 1st comp $\left.\begin{array}{l} c=7 \\ c=0 \end{array}\right\}$ $\boxed{\text{Nope.}}$

② Yes, $-2(1,2,6,-1) = (-2,-4,-12,2)$

Related to this was the homework question:

(7.21) Find all pts $(x,y,z)$ where the surface $z = x e^{-x^2-y^2}$ is parallel to the xy plane.

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Method 1: $z = f(x,y)$ graph of a fcn.

We want to know where the surface graph has a horizontal tangent plane.

(∴ parallel to $z=0$ ← xy plane)

So we want $f_x = 0$

$f_y = 0$

$$z_x = e^{-x^2-y^2} + x(-2x)e^{-x^2-y^2} = 0$$

$$(1-2x^2)\, e^{-x^2-y^2} = 0 \quad (*).$$

$$z_y = -2xy e^{-x^2-y^2} = 0. \quad (**).$$

Note: exponential fcns are always positive, so we can divide both sides by those.

$(*)$ $\quad 1-2x^2 = (1-\sqrt{2}\,x)(1+\sqrt{2}\,x) = 0.$

$(**)$ $\quad -2xy = 0.$

$(*) \Rightarrow \quad x = \frac{1}{\sqrt{2}}$ or $x = -\frac{1}{\sqrt{2}}$ ] True

$(**) \Rightarrow \quad \underbrace{x = 0}_{\text{impossible due to } (*)}$ or $y = 0$ ✓ ] True

$x = \frac{1}{\sqrt{2}}, \; y = 0, \; z = x e^{-x^2-y^2} = \frac{1}{\sqrt{2}} e^{-1/2}$

or $x = -\frac{1}{\sqrt{2}}, \; y = 0, \; z = -\frac{1}{\sqrt{2}} e^{-1/2}$

points $\boxed{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} e^{-1/2}\right) \text{ and } \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} e^{-1/2}\right)}$

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## Method 2

$z = x e^{-x^2-y^2}$

works even if equation is not $z = f(x,y)$

$F(x,y,z) = x e^{-x^2-y^2} - z = 0$

a level set (contour)

The tangent plane at each $(x,y,z)$ is perpendicular to the $\nabla F$.

$xy$ plane — normal vector $= (0,0,1)$

$\boxed{\begin{array}{l} z = 0 \\ \nabla(z) = (0,0,1) \end{array}}$

We want $\nabla F = c(0,0,1)$

some constant.

$$\nabla F = (F_x, F_y, F_z) = \left(e^{-x^2-y^2} - 2x^2 e^{-x^2-y^2}, \; -2xy e^{-x^2-y^2}, \; -1\right)$$

$$\Rightarrow \vec{\nabla} F = \left( (1-2x^2)e^{-x^2-y^2}, -2xye^{-x^2-y^2}, -1 \right) = c(0,0,1)$$

$$\Rightarrow \left.\begin{aligned} (1-2x^2)e^{-x^2-y^2} &= 0 \\ -2xye^{-x^2-y^2} &= 0 \end{aligned}\right] \text{same analysis as before}$$

$$(-1) = c(1) \rightarrow c=-1.$$

$$x = \pm \frac{1}{\sqrt{2}}$$

$$y = 0$$

$$z = xe^{-x^2-y^2} = \pm\frac{1}{\sqrt{2}}e^{-1/2}.$$

$$\Rightarrow \boxed{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}e^{-1/2}\right), \left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}e^{-1/2}\right).}$$

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Related question I didn't ask:

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<u>Example</u> Find all points where the surface $z = xe^{-x^2-y^2}$ has a tangent plane parallel to $-x+y+4z=12$.

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Let $F(x,y,z) = xe^{-x^2-y^2} - z = 0$
level set.

$$\nabla F = c\,\nabla(-x+y+4z) = c(-1,1,4)$$

$$\left((1-2x^2)e^{-x^2-y^2},\ -2xy\,e^{-x^2-y^2},\ -1\right) = c(-1,1,4)$$

$$\begin{cases}(1-2x^2)e^{-x^2-y^2} = -c \\ -2xy\,e^{-x^2-y^2} = c \\ -1 = 4c\end{cases}$$

3 equations
3 unknowns

$\hookrightarrow c = \frac{-1}{4}$.

$$\Rightarrow \begin{cases}(1-2x^2)e^{-x^2-y^2} = \frac{1}{4} \\ -2xy\,e^{-x^2-y^2} = \frac{-1}{4}\end{cases}$$

Now
2 eqns
2 unknowns.

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Add: $(1-2x^2-2xy)\underbrace{e^{-x^2-y^2}}_{\text{positive (divide by it)}} = 0$

$$\Rightarrow (1-2x^2-2xy) = 0$$

We can now solve for $x$ or $y$ ← eliminate one more variable.

$$1-2x^2 = 2xy$$

$$\frac{1}{2x} - x = y$$

Note: Can $x$ be zero? No: 2nd eqn would be $0 = \frac{-1}{4}$.

We now plug $y$ into the first eqn.:

$$(1-2x^2)e^{-x^2-\left(\frac{1}{2x}-x\right)^2} = \frac{1}{4}$$

$$(1-2x^2)e^{-x^2-\left(\frac{1}{4x^2}-1+x^2\right)} = \frac{1}{4}$$

$$\Rightarrow (1-2x^2)\,e^{\left(-2x^2-\frac{1}{4x^2}+1\right)} = \frac{1}{4}.$$ 1 eqn 1 unknown

Computer: solutions are

$$x \simeq 0.376,\ 0.555,\ -0.376,\ -0.555$$

$$y = \frac{1}{2x}-x = 0.954,\ 0.346,\ -0.954,\ -0.346$$

$$z = xe^{-x^2-y^2} = .131,\ .391,\ -.131,\ -.391$$

Answer: $(.376, .954, .131) = A$

$(.555, .346, .391) = B$

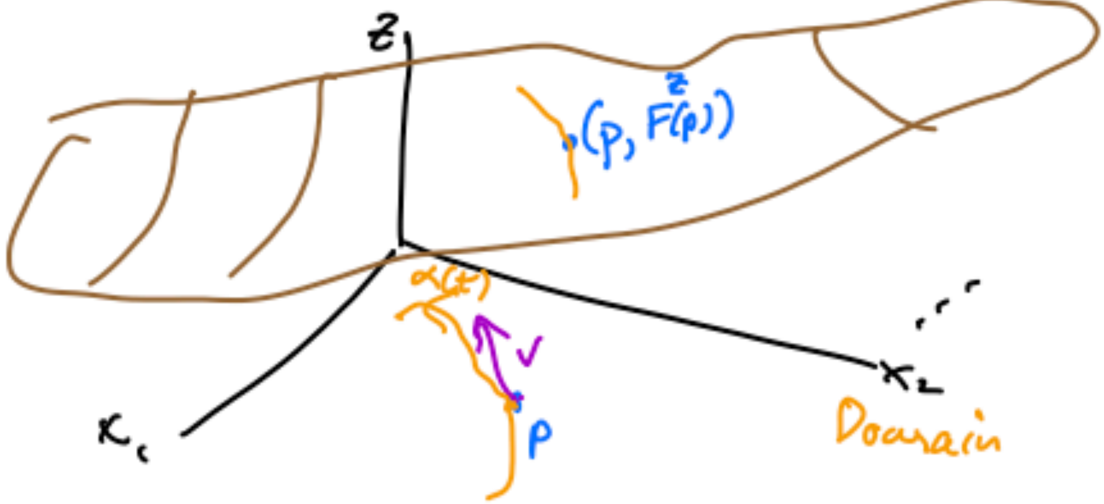
also $-A$, $-B$.

New kind of derivative for functions of several variables (Mainly concentrate on Real-Valued Functions.):

Directional Derivative

Picture $F: \mathbb{R}^n \to \mathbb{R}$


Graph $z = F(x_1, x_2, \ldots)$



Given $z = F(x_1, x_2, \ldots)$

point $p = (p_1, p_2, \ldots)$ in domain

vector $v = (v_1, v_2, \ldots)$ in domain

$\alpha(t)$ curve

$$\left.\begin{aligned} \alpha(0) &= p \\ \alpha'(0) &= v \end{aligned}\right\} \quad \begin{aligned} &\text{Example} \\ &\alpha(t) = p + tv \end{aligned}$$

$$\begin{pmatrix} \text{Directional} \\ \text{Derivative} \\ \text{of } F \\ \text{with respect} \\ \text{to } v \text{ at } p \end{pmatrix} = \frac{d}{dt}\Big(F(\alpha(t))\Big)\Big|_{t=0}$$

$$= \frac{d}{dt}\Big(F(p+tv)\Big)\Big|_{t=0}$$

gives the rate of change of F when you go with velocity v away from p.

Example : $F(x,y)$

$v = (1,0)$

$$\frac{\partial F}{\partial v} = \underset{=v(F)}{D_v F} = \frac{d}{dt}\Big(F(p+tv)\Big)\Big|_{t=0} = F_x$$

$v = (0,1)$

$$\frac{\partial F}{\partial v} = \frac{d}{dt}\left(F(p+tv)\right)\Big|_{t=0} = F_y$$

Using the chain rule:

$$\frac{d}{dt}\left(F(p+tv)\right) = F'(p+tv)\cdot v\,\Big|_{t=0}$$

$$= \nabla F(p)\cdot v$$

to calculate: $\boxed{\dfrac{\partial F}{\partial v} = \nabla F \cdot v}$ ☆